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Exact Solutions for Self-Dual SU(2) and SU(3) Yang–Mills Fields

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The (constrained) canonical reduction of four-dimensional self-dual SU(2) and SU(3) Yang–Mills theory to two-dimensional nonlinear Schrödinger (NS) and Korteweg–de Vries (KdV) equations are considered. The Bäcklund transformations (BTs) are implemented to obtain new classes of exact solutions for the reduced two-dimensional NS and KdV models.

KEY WORDS: Self-dual SU(2); self-dual SU(3); Yang–Mills fields.

1. INTRODUCTION

The self-dual Yang–Mills (SDYM) equations were introduced by Yang and Mills (1954). The essential idea of Yang and Mills (1954) is to consider an analytic continuation of the gauge potential A_μ into a complex space where x_1, x_2, x_3, and x_4 are complex. The self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ are then valid also in complex space, in a region containing real space where the xs are real. Now consider four new complex variables y, $\bar{y}$, z, and $\bar{z}$ defined by

$$\sqrt{2}y = x_1 + ix_2, \qquad \sqrt{2}\bar{y} = x_1 - ix_2,$$
$$\sqrt{2}z = x_3 - ix_4, \qquad \sqrt{2}\bar{z} = x_3 + ix_4. \tag{1.1}$$

The canonical formalism for the SDYM system has been established by Chau and Yamanaka (1992, 1993). It is simple to check that the self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduce to

$$F_{yz} = 0, \qquad F_{\bar{y}\bar{z}} = 0,$$
$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{1.2}$$

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Equations (1.2) can be immediately integrated, since they are pure gauge, to give

$$A_y = D^{-1} D_y, \quad A_z = D^{-1} D_z, \quad A_{\bar{y}} = \bar{D}^{-1} \bar{D}_{\bar{y}}, \quad A_{\bar{z}} = \bar{D}^{-1} \bar{D}_{\bar{z}}, \tag{1.3}$$

where D and $\bar{D}$ are arbitrary 2×2 complex matrix functions of y, $\bar{y}$, z, and $\bar{z}$ with determinant $= 1$ (for SU(2) gauge group) and D and $\bar{D}$ are arbitrary 3×3 complex matrix functions of y, $\bar{y}$, z, and $\bar{z}$ with determinant $= 1$ (for SU(3) gauge group), $D_y = \partial_y D$, etc.

For real gauge fields $A \doteq -A^+$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3, and x_4), we require

$$\bar{D} \doteq (D^+)^{-1}. \tag{1.4}$$

Gauge transformations are the transformations

$$D \to DU, \quad \bar{D} \to \bar{D}U, \quad U^+ U \doteq I \tag{1.5}$$

where U is a 2×2 complex matrix function of y, $\bar{y}$, z, and $\bar{z}$ with determinant $= 1$ (for SU(2) gauge group) and U is a 3×3 complex matrix function of y, $\bar{y}$, z, and $\bar{z}$ with determinant $= 1$ (for SU(3) gauge group). Under transformation (1.5), Eq. (1.4) remains unchanged. We now define the Hermitian matrix J as

$$J \equiv D\bar{D}^{-1} \doteq DD^+. \tag{1.6}$$

J has the very important property of being invariant under the gauge transformation equation (1.5). The only nonvanishing field strengths in terms of J become

$$F_{u\bar{v}} = -\bar{D}^{-1} (J^{-1} J_u)_{\bar{v}} \bar{D}, \tag{1.7}$$

$(u, v = y, z)$ and the remaining self-duality Eq. (1.2) takes the form:

$$(J^{-1} J_y)_{\bar{y}} + (J^{-1} J_z)_{\bar{z}} = 0. \tag{1.8}$$

The action density in terms of J is

$$\begin{aligned} \Phi(J) &= -\frac{1}{2} \operatorname{Tr} F_{\mu\nu} F_{\mu\nu} = -2 \operatorname{Tr}(F_{y\bar{y}} F_{z\bar{z}} + F_{y\bar{z}} F_{\bar{y}z}) \\ &= -2 \operatorname{Tr}\{(J^{-1} J_y)_{\bar{y}} (J^{-1} J_z)_{\bar{z}} - (J^{-1} J_y)_{\bar{z}} (J^{-1} J_z)_{\bar{y}}\} \end{aligned} \tag{1.9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, \ A_\nu] \tag{1.10}$$

In this paper, we present a set of exact solutions by applying the BTs (Abolwitz and Clarkson, 1991; Chan and Zheng, 1989; Rogers and Shodwick, 1982) for NS and KdV equations in two dimensions. Consequently we find exact solutions for self-dual SU(2) and SU(3) Yang–Mills equations. The paper is organized as follows: This introduction is followed by the reduction of self-dual SU(2) Yang–Mills theory to nonlinear Schrödinger (NS) and Korteweg–de Vries (KdV) theories in two dimensions, together with the implementation of Bäcklund transformations (BTs) to generate new classes of exact solutions in Section 2. In Section 3 the reduction of self-dual SU(3) Yang–Mills theory to NS and KdV theories in two dimensions as well.

2. EXACT SOLUTIONS FOR SELF-DUAL SU(2) YANG–MILLS EQUATIONS

2.1. Canonical Reduction of Self-Dual SU(2) Yang–Mills Theory to Nonlinear Schrödinger NS Equation in Two Dimensions and Exact Solutions

It is well-known that a two-dimensional integrable NS equation can be obtained by reduction from (1.2) and (1.3), i.e. the NS equation can be obtained by putting (Ge *et al.*, 1994; Khater *et al.*, 1999a)

$$A_y = aI_3, \quad A_z = \frac{1}{2} i I_1, \quad A_{\bar{z}} = bI_2, \quad A_{\bar{y}} = 0, \tag{2.1.1}$$

where a and b depend on y and $\bar{y}$ only, and I_i are pauli matrices

$$I_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with $U = U(y, \bar{y})$ and

$$a = U_{\bar{y}}, \qquad b = -iU_y - 2U^2U^*. \tag{2.1.2}$$

Substituting (2.1.1) and (2.1.2) into (1.2), we obtain the NS equation

$$iU_y + U_{\bar{y}\bar{y}} + 2U^2U^* = 0. \tag{2.1.3}$$

Then we find the solution of NS equation by the AKNS (Ablowitz, Kaup, Newell, and Segur) system as follows (Ablowitz *et al.*, 1973):

It is known that many nonlinear evolution equations (NEEs) can be derived from the following AKNS system

$$\phi_{\bar{y}} = P\phi, \qquad \phi_y = Q\phi, \tag{2.1.4}$$

where

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.1.5}$$

and P and Q are two 2×2 null-trace matrices,

$$P = \begin{bmatrix} \eta & q \\ r & -n \end{bmatrix}, \qquad Q = \begin{bmatrix} A & B \\ C & -A \end{bmatrix}. \tag{2.1.6}$$

Here η is a parameter, independent of y and $\bar{y}$, and q and r are functions of y and $\bar{y}$, where P and Q must satisfy the following integrability condition:

$$P_y - Q_{\bar{y}} + PQ - QP = 0. \tag{2.1.7}$$

We obtain various NEEs of physical interest depending on the choices of A, B, C, and r. Konno and Wadati (1975) introduced the function

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{2.1.8}$$

and, for each of the NEE, derived a BT with the following form:

$$U' = U + f(\Gamma,\ \eta), \tag{2.1.9}$$

where U' is a new solution.

To construct the NS equation (2.1.3) from the AKNS system (2.1.4), we take r, A, B, and C in the form

$$\begin{aligned} r &= -U^*, \quad q = U, \quad A = 2i\eta^2 + iUU^*, \\ B &= 2i\eta U + iU_{\bar{y}}, \quad C = -2i\eta U^* + iU^*_{\bar{y}} \end{aligned} \tag{2.1.10}$$

Substituting from Eq. (2.1.10) into the condition (2.1.7), we get the NS equation (2.1.3), and derive the new solution U' from the known solution U by using the BT

$$U' = -U - \frac{4\eta\Gamma}{1 + |\Gamma|^2}. \tag{2.1.11}$$

Now, we shall choose a known solution of the above NS equation (2.1.3). Next we solve the AKNS system (2.1.4) for ϕ_1 and ϕ_2. Then by Eq. (2.1.8) and the corresponding BT (2.1.11) we will obtain a new solution of the NS equation (2.1.3). Let the constant solution of (2.1.3) be

$$U = 0. \tag{2.1.12}$$

From the AKNS system (2.1.4),

$$d\phi = \phi_y\, dy + \phi_{\bar{y}}\, d\bar{y} = P\phi\, d\rho, \tag{2.1.13}$$

where

$$P = \begin{bmatrix} \eta & 0 \\ 0 & -\eta \end{bmatrix}, \qquad (\eta \text{ is real}),$$

$$\rho = \bar{y} + 2i\eta y. \tag{2.1.14}$$

The solution of (2.1.13) is

$$\phi = \begin{bmatrix} \exp(\eta\rho) & 0 \\ 0 & \exp(\eta\rho) \end{bmatrix} \phi_0. \tag{2.1.15}$$

Now, we choose $\varphi_0 = (1,\ 1)^T$ in (2.1.15) and use (2.1.8), then BT (2.1.11) gives the new solutions of the NS equation (2.1.3) corresponding to the known constant NS solutions (2.1.12)

$$U = -2\eta\ \exp(4i\eta^2 y)\ \text{sech}(2\eta\bar{y}). \tag{2.1.16}$$

Then we find the constants a and b. Consequently we obtain the gauge potential A_μ.

$$a = U_{\bar{y}} = -4\eta^2 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) \tanh(2\eta\bar{y}), \tag{2.1.17}$$

$$b = -8\eta^3 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) + 16\eta^3 \exp(4i\eta^2(2y - \bar{y})) \times (\text{sech}^2 2\eta\bar{y})(\text{sech} 2\eta y). \tag{2.1.18}$$

Then we find

$$A_y = \begin{bmatrix} -4\eta^2 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) \times \tanh(2\eta\bar{y}) & 0 \\ 0 & 4\eta^2 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) \times \tanh(2\eta\bar{y}) \end{bmatrix},$$

$$A_{\bar{y}} = 0, \tag{2.1.19}$$

$$A_z = \begin{bmatrix} 0 & (1/2)i \\ (1/2)i & 0 \end{bmatrix}, \tag{2.1.20}$$

$$A_{\bar{z}} = \begin{bmatrix} 0 & -i[-8\eta^3 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) + 16\eta^3 \exp(4i\eta^2(2y - \bar{y})) \times (\text{sech}^2 2\eta\bar{y})(\text{sech} 2\eta y)] \\ i[-8\eta^3 \exp(4i\eta^2 y)(\text{sech} 2\eta\bar{y}) + 16\eta^3 \exp(4i\eta^2(2y - \bar{y})) \times (\text{sech}^2 2\eta\bar{y})(\text{sech} 2\eta y)] & 0 \end{bmatrix} \tag{2.1.21}$$

2.2. Canonical Reduction of Self-Dual SU(2) Yang–Mills Theory to KdV Equation in Two Dimensions and Exact Solutions

It is well-known that the two-dimensional integrable KdV equation can be obtained by reduction from (1.2) and (1.3), then the KdV equation can be obtained by setting (Ge *et al.*, 1994; Khater *et al.*, 1999a)

$$A_y = \mathrm{a} I_3, \quad A_{\bar{y}} = 0, \quad A_z = \mathrm{b} I_2, \quad A_{\bar{z}} = \frac{1}{2} i I_1, \tag{2.2.1}$$

where

$$a = q_{\bar{y}\bar{y}} + 3q^2, \qquad b = q_y.$$

Substituting (2.2.1) into (1.2), we obtain the KdV equation

$$q_y + q_{\bar{y}\bar{y}\bar{y}} + 6qq_{\bar{y}} = 0 \tag{2.2.2}$$

To construct the KdV equation (2.2.2) from the AKNS system (2.1.4), we take P and Q in the form

$$P = \begin{bmatrix} \eta & q \\ -1 & -\eta \end{bmatrix},$$
$$Q = \begin{bmatrix} -4\eta^3 - 2\eta q - q_{\bar{y}} & -4\eta^2 q - 2q^2 - 2\eta q_{\bar{y}} - q_{\bar{y}\bar{y}} \\ 4\eta^2 + 2q & 4\eta^3 + 2\eta q + q_{\bar{y}} \end{bmatrix}. \tag{2.2.3}$$

Substituting from (2.2.3) into the condition (2.1.7), we get the KdV equations (2.2.1), and to derive the new solution q' from the known solution q by using the BT

$$q' = q - 2 \ln \Gamma_{\bar{y}}. \tag{2.2.4}$$

Now we shall choose a known solution of the above KdV equation (2.2.2) as a simple function $q = q(y, \bar{y})$ and substitute these solutions into the corresponding matrices P and Q. By direct calculation we take

$$q = \frac{\bar{y} - k_1}{3(2y - k_2)} \quad \left(k_1 \text{ and } k_2 \text{ are constants, } y \neq \frac{k_2}{2} \right) \tag{2.2.5}$$

as a simple function solution of the KdV equation (2.2.2).

In this case the system (2.1.4)–(2.1.6) cannot be solved for the vector ϕ as a whole, but can be solved in component forms ϕ_1 and ϕ_2 separately. From (2.1.4)–(2.1.6), after inserting the known solution $q(y, \bar{y})$ of the KdV equation into the corresponding matrices P and Q, one has the following system of partial differential equations for the unknowns ϕ_1 and ϕ_2:

$$\phi_{\bar{y}} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{\bar{y}} = P\phi = \begin{bmatrix} \eta & q \\ r & -\eta \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

and

$$\phi_y = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_y = Q\phi = \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

Hence

$$\phi_{1\bar{y}} = \eta\phi_1 + q\phi_2, \tag{2.2.6}$$

$$\phi_{2\bar{y}} = r\phi_1 + \eta\phi_2, \tag{2.2.7}$$

$$\phi_{1y} = A\phi_1 + B\phi_2, \tag{2.2.8}$$

and

$$\phi_{2y} = C\phi_1 + A\phi_2. \tag{2.2.9}$$

These equations are compatible under the conditions of the assumed values of matrices P and Q connected with the considered NEEs. Solve ϕ_1 from (2.2.7) giving

$$\phi_1 = \left(\frac{1}{r}\right)(\phi_{2\bar{y}} + \eta\phi_2). \tag{2.2.10}$$

Substituting ϕ_1 into (2.2.9)

$$\phi_{2y} = \left(\frac{C}{r}\right)(\phi_{2\bar{y}} + \eta\phi_2) - A\phi_2,$$

and by using Eq. (2.1.7) yields

$$r_y - C_{\bar{y}} + 2rA - 2\eta C = 0,$$

$$r\phi_{2y} = C(\phi_{2\bar{y}} + \eta\phi_2) - rA\phi_2,$$

then

$$C\phi_{2\bar{y}} - r\phi_{2y} = \left(\frac{1}{2}\right)(C_{\bar{y}} - r_y)\phi_2. \tag{2.2.11}$$

This is a linear first-order partial differential equation with ϕ_2 as its unknown function; it can be solved by the method of characteristics. After ϕ_2 has been obtained from (2.2.11); substituting it into (2.2.10), one obtains ϕ_1 and thus the resulting two general solutions ϕ_1 and ϕ_2, which contain an arbitrary function g. This arbitrary function can be determined by demanding that the two solutions ϕ_1 and ϕ_2 satisfy either (2.2.6) or (2.2.8), which will yield a second-order linear ordinary differential equation with the function g as its unknown. If one can solve for the function g, we obtain the two particular solutions ϕ_1 and ϕ_2. Finally, by applying (2.1.8) and the BT corresponding to the KdV equation we shall obtain a new solution of the KdV equation. Inserting (2.2.5) into (2.2.11), together with (2.2.3), gives

$$4\eta^2 + \frac{2(\bar{y} - k_1)}{3(2y - k_2)}\phi_{2\bar{y}} + \phi_{2y} = \frac{1}{3(2y - k_2)}\phi_2. \tag{2.2.12}$$

Equation (2.2.12) has the following system of ordinary differential equations as its characteristic equations,

$$\frac{d\bar{y}}{dy} = 4\eta^2 + \frac{2(\bar{y} - k_1)}{3(2y - k_2)}, \tag{2.2.13}$$

$$\frac{d\phi_2}{dy} = \frac{1}{3(2y - k_2)}. \tag{2.2.14}$$

Solving these two equations gives the general solution of the unknown ϕ_2 in Eq. (2.2.12), which reads

$$\phi_2 = (2y - k_2)^{1/6} g(\theta),$$
$$\theta = (\bar{y} - k_1)(2y - k_2)^{-1/3} - 3\eta^2(2y - k_2)^{2/3}, \tag{2.2.15}$$

where g is an arbitrary differentiable function. Substituting (2.2.5) and (2.2.15) into (2.2.10) gives the general solution of ϕ_1, which reads

$$\phi_1 = -(2y - k_2)^{-1/6} \frac{dg}{d\theta} - \eta(2y - k_2)^{1/6} g(\theta). \tag{2.2.16}$$

To determine the function $g(\theta)$, substituting (2.2.5), (2.2.15), and (2.2.16) into (2.2.6), we find that $g(\theta)$ must satisfy the following Airy equation (Oliver, 1974):

$$\frac{d^2 g}{d\theta^2} + \frac{\theta}{3} g = 0. \tag{2.2.17}$$

i.e.

$$g = (A_1 C_1 + B_1 C_2)\left(i\theta \left(\frac{-1}{3}\right)^{1/3}\right), \tag{2.2.18}$$

where A_1 and B_1 are two Airy functions,

$$A_1 = A_1(\theta), \tag{2.2.19}$$

$$B_1 = B_1(\theta), \tag{2.2.20}$$

where C_1 and C_2 are two arbitrary constants. After g has been determined, (2.2.15), (2.2.16), and (2.1.8) lead to

$$\Gamma = -(2y - k_2)^{-1/3} \frac{d}{d\theta}(\ln g) - \eta; \tag{2.2.21}$$

then substituting this Γ and (2.2.5) into the BT (2.2.4), we arrive at the new solution q' of the KdV equation (2.2.2) corresponding to the known solution (2.2.5):

$$q' = \frac{\bar{y} - k_1}{3(2y - k_2)} + 2(2y - k_2)^{-2/3} \frac{d^2}{d\theta^2}(\ln g), \tag{2.2.22}$$

where

$$\theta = (\bar{y} - k_1)(2y - k_2)^{-1/3} - 3\eta^2(2y - k_2)^{2/3}.$$

Consequently we can calculate the gauge potential A_μ, from Eq. (2.2.22). Then

$$A_y = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad A_{\bar{y}} = 0, \quad A_{\bar{z}} = \begin{bmatrix} 0 & (1/2)i \\ (1/2)i & 0 \end{bmatrix}$$

$$A_z = \begin{bmatrix} 0 & -ib \\ ib & 0 \end{bmatrix}, \tag{2.2.23}$$

where

$$a = 2(2y - k_2)^{-4/3}\frac{d^4}{d\theta^4}(\ln g) + 3\left[\frac{(\bar{y} - k_1)}{3(2y - k_2)} + 2(2y - k_2)^{-3/2}\frac{d^2}{d\theta^2}(\ln g)\right]^2$$

$$(2.2.24)$$

$$b = \frac{-2(\bar{y} - k_1)}{3(2y - k_2)^2}\left[1 + 2\frac{d^3}{d\theta^3}(\ln g)\right] - 8(2y - k_2)^{-1} \times \left[\frac{1}{3}(2y - k_2)^{-2/3}\frac{d^2}{d\theta^2}(\ln g) + \eta^2\frac{d^3}{d\theta^3}(\ln g)\right].$$

We note that the self-dual SU(2) Yang–Mills equations hold.

3. EXACT SOLUTIONS FOR SELF-DUAL SU(3) YANG–MILLS EQUATIONS

3.1. Canonical Reduction of Self-Dual SU(3) Yang–Mills Theory to NS Equation in Two Dimensions and Exact Solutions

It is well-known that two-dimensional integrable sine-Gordon equation can be obtained by reduction from (1.2) and (1.3); then the NS equation can be obtained by setting (Khater *et al.*, 1999b)

$$\begin{aligned} A_y &= (-ai)\lambda_2, \qquad A_{\bar{y}} = 0, \\ A_z &= \frac{1}{4}\lambda_1, \qquad A_{\bar{z}} = b\lambda_3, \end{aligned} \tag{3.1.1}$$

where a and b depend on y and $\bar{y}$ only, and λ_a ($a = 1, \ldots, 8$) are the SU(3) Gell–Mann matrices, with $U = U(y, \bar{y})$ and

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$a = U_{\bar{y}}, \qquad b = -iU_y - 2U^2U^*. \tag{3.1.2}$$

Substituting (3.1.1) and (3.1.2) into (1.2), we obtain the NS equation

$$iU_y + U_{\bar{y}\bar{y}} + 2U^2U^* = 0. \tag{3.1.3}$$

From Eq. (2.1.16) we obtain the gauge potential A_μ

$$A_y = \begin{bmatrix} 0 & 0 & -4\eta^2 \exp(4i\eta^2 y) \times (\operatorname{sech} 2\eta\bar{y}) \tanh(2\eta\bar{y}) \\ 0 & 0 & 0 \\ -4\eta^2 \exp(4i\eta^2 y) \times (\operatorname{sech} 2\eta\bar{y}) \tanh(2\eta\bar{y}) & 0 & 0 \end{bmatrix},$$

$$A_{\bar{y}} = 0, \tag{3.1.4}$$

$$A_z = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 0 & 0 & 0 \\ -1/4 & 0 & -1/4 \end{bmatrix}, \tag{3.1.5}$$

$$A_{\bar{z}} = \begin{bmatrix} -8\eta^3 \exp(4i\eta^2 y)(\operatorname{sech} 2\eta\bar{y}) + 16\eta^3 \exp(4i\eta^2(2y-\bar{y})) \times (\operatorname{sech}^2 2\eta\bar{y})(\operatorname{sech} 2\eta y) & 0 & 8\eta^3 \exp(4i\eta^2 y)(\operatorname{sech} 2\eta\bar{y}) - 16\eta^3 \exp(4\imath\eta^2(2y-\bar{y})) \times (\operatorname{sech}^2 2\eta\bar{y})(\operatorname{sech} 2\eta y) \\ 0 & 0 & 0 \\ -8\eta^3 \exp(4i\eta^2 y)(\operatorname{sech} 2\eta\bar{y}) + 16\eta^3 \exp(4i\eta^2(2y-\bar{y})) \times (\operatorname{sech}^2 2\eta\bar{y})(\operatorname{sech} 2\eta y) & 0 & 8\eta^3 \exp(4i\eta^2 y)(\operatorname{sech} 2\eta\bar{y}) - 16\eta^3 \exp(4i\eta^2(2y-\bar{y})) \times (\operatorname{sech}^2 2\eta\bar{y})(\operatorname{sech} 2\eta y) \end{bmatrix} \tag{3.1.6}$$

3.2. Canonical Reduction of Self-Dual SU(3) Yang–Mills Theory to KdV Equation in Two Dimensions and Exact Solutions

It is well-known that the two-dimensional integrable KdV equation can be obtained by reduction from (1.2) and (1.3), then the KdV equation can be obtained by setting (Khater *et al.*, 1999b)

$$A_y = ai\lambda_2, \quad A_{\bar{y}} = 0, \quad A_z = -(1/4)\lambda_1, \quad A_{\bar{z}} = b\lambda_3, \tag{3.2.1}$$

where

$$a = q_{\bar{y}\bar{y}} + 3q^2, \qquad b = q_y.$$

Substituting (3.2.1) into (1.2), we obtain the KdV equation

$$q_y + q_{\bar{y}\bar{y}\bar{y}} + 6qq_{\bar{y}} = 0 \tag{3.2.2}$$

Consequently, we can calculate the gauge potential A_μ, from Eq. (2.2.22), then

$$A_y = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}, \quad A_{\bar{y}} = 0, \quad A_{\bar{z}} = \begin{bmatrix} b & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & -b \end{bmatrix}$$

$$A_z = \begin{bmatrix} -(1/4) & 0 & -(1/4) \\ 0 & 0 & 0 \\ 1/4 & 0 & 1/4 \end{bmatrix}. \tag{3.2.3}$$

where

$$a = 2(2y - k_2)^{-4/3}\frac{d^4}{d\theta^4}(\ln g) + 3\left[\frac{(\bar{y} - k_1)}{3(2y - k_2)} + 2(2y - k_2)^{-3/2}\frac{d^2}{d\theta^2}(\ln g)\right]^2 \tag{3.2.4}$$

$$b = \frac{-2(\bar{y} - k_1)}{3(2y - k_2)^2}\left[1 + 2\frac{d^3}{d\theta^3}(\ln g)\right] - 8(2y - k_2)^{-1} \times \left[\frac{1}{3}(2y - k_2)^{-2/3}\frac{d^2}{d\theta^2}(\ln g) + \eta^2\frac{d^3}{d\theta^3}(\ln g)\right].$$

We note that the self-dual SU(3) Yang–Mills equations hold.

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